3D Projection

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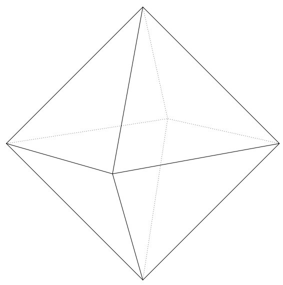
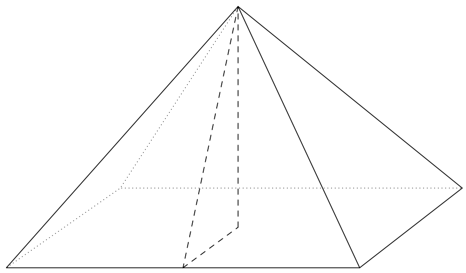
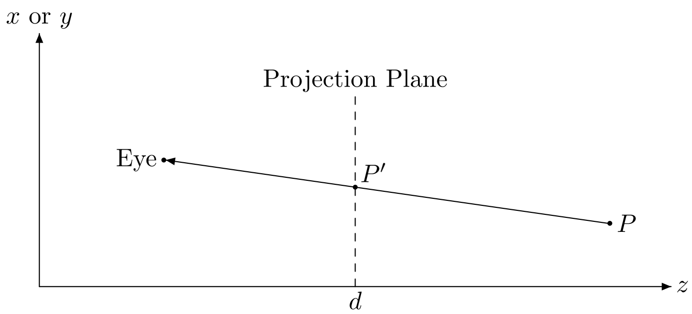
To some extend, 3-dimensional objects can be rendered onto a plane in such a way that the resulting image appears to have depth:





Let $(a, b, c) \in \mathbb{R}^3$ be an arbitrary point in 3-dimensional space that denotes the position of the eye. Let $P = (x, y, z)$ be a point that we would like to project onto a 2-dimensional plane that is located between P and the eye at distance d.



The line of sight hits the projection plane at a specific point $P' = (x', y', d)$, and this intersection determines the coordinate (x', y') that represents P in 2-dimensional space. The line of sight can be represented as $\text{Eye} + t\,(P - \text{Eye})$ for $t \in [0, 1]$. It follows that the projection P' has the following coordinates:

$$\begin{aligned} x' &= a + t\,(x - a) \\ y' &= b + t\,(y - b) \\ z' &= c + t\,(z - c). \end{aligned}$$

Using the known fact $z' = d$, the value of t can be derived:

$$d = c + t\,(z - c) \iff t = \frac{d - c}{z - c}.$$

It follows that 3D projection is a linear transformation $p\colon \mathbb{R}^3 \to \mathbb{R}^2$,

$$p \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a + \frac{d-c}{z-c}\,(x-a) \\ b + \frac{d-c}{z-c}\,(y-b) \end{pmatrix}.$$

The 2-dimensional projection of 3-dimensional space exhibits a phenomenon known as a "vanishing point". Points that are far away on the z axis converge to the projection of the eye:

$$\lim_{z \to \infty} p \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

The vanishing effect becomes more pronounced, when the distance between eye and projection plane decreases. When that distance increases, however, the distortion becomes less visible until it effectively disappears:

$$\lim_{c \to -\infty} p \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The following images both show the same point P, but in the second rendering the eye has moved closer to the projection plane. As a result, all projected points have moved closer to the vanishing point.