What is the distance to the horizon?

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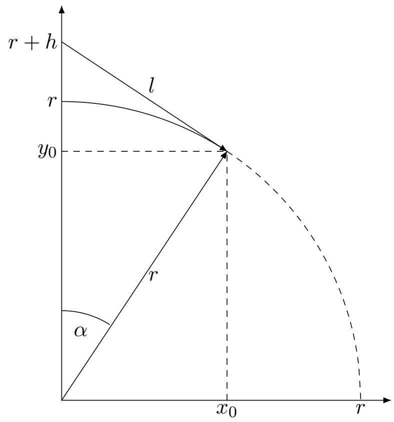
2010-11-19

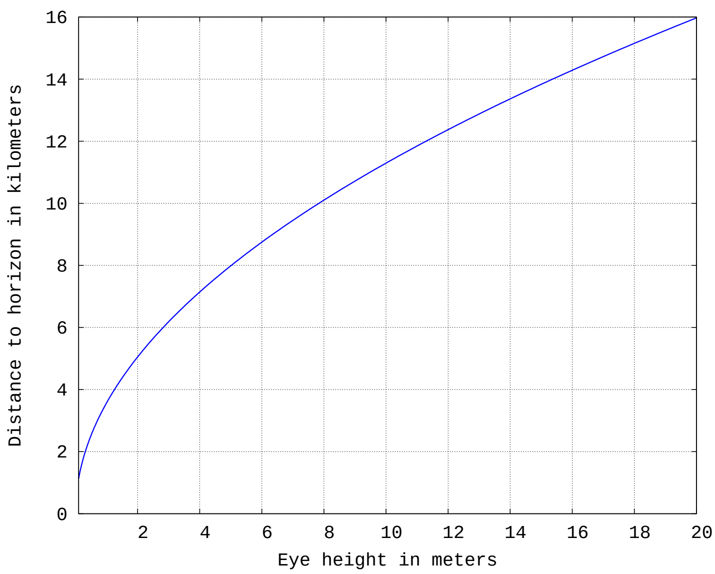


The distance to the horizon can be approximated fairly accurately by Euclidean geometry. Earth is assumed to be a sphere of radius $r = 6378$ kilometers. Let h be the eye height of an observer. Any ray of light that hits the eye represents one of infinitely many potential lines of sight. Now, of particular interest is the one line of sight that is a tangent to the Earth's surface. Tangent l intersects with the surface in exactly one point, and any of the Earth's surface beyond that point is invisible at that eye height.

Tangent l is by definition perpendicular to the Earth radius. Pythagoras' theorem states $(r+h)^2 = r^2 + l^2$, and by solving for l the distance is obtained:

$$l = \sqrt{2\,h\,r + h^2}.$$

The following graphs shows how l develops in relation to h:



Interestingly enough, the first diagram shows another right triangle spanned with hypotenuse l, for which Pythagoras' theorem states that

$$l = \sqrt{{x_0}^2 + (r + h - y_0)^2} = \sqrt{2\,h\,r + h^2}$$

must hold. In order to verify this assertion, however, the intersection point (x_0, y_0) has to be determined. Earth's surface is a function

$$\begin{aligned} f &\colon [0, r] \to [0, r] \\ f(x) &= \sqrt{r^2 - x^2}. \end{aligned}$$

At the point where f and l intersect, the slope of l equals the slope of f. Using the derivative

$$f'(x) = -\frac{x}{\sqrt{r^2 - x^2}},$$

an explicit representation of tangent l is the function

$$l(x) = f(x_0) + f'(x_0)\,(x - x_0).$$

By definition, $l(0) = r + h$ must hold, because the eye of the observer is another point of l. Solving the equation for x_0 is fairly straightforward:

$$\begin{aligned} & f(x_0) + f'(x_0)\,(0 - x_0) = r + h \\ \iff \quad & \frac{r^2}{\sqrt{r^2 - {x_0}^2}} = r + h \\ \iff \quad & x_0 = \frac{r}{r + h}\,\sqrt{2\,h\,r + h^2}. \end{aligned}$$

The expected result $l = \sqrt{2\,h\,r + h^2}$ follows if x_0 is substituted into the equation given by Pythagoras' theorem, which in turn allows further simplification of the point of intersection to

$$x_0 = \frac{r\,l}{r + h}.$$

So far, the distance to the horizon was regarded as the length of a line of sight, but that length is not the same as the distance that a person walking Earth's surface from $(0, r)$ to (x_0, y_0) would travel. That distance would be $r\,\alpha$. The angle α is defined as

$$\sin(\alpha) = \frac{x_0}{r} \quad \text{or} \quad \cos(\alpha) = \frac{r}{r + h},$$

and solving for α gives:

$$\alpha = \arcsin\left(\frac{l}{r + h}\right) = \arccos\left(\frac{r}{r + h}\right).$$

It turns out that the difference between l and $r\,\alpha$ is negligible. Even if the eye of the observer is as high as 10,000 meters above the ground, l and $r\,\alpha$ differ by less than 400 meters — which is hardly significant, given the magnitude of error involved in the computation to begin with just because Earth is not actually a sphere.