Properties of Hierarchical Structures

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1 Theory

Definition. *A hierarchy is a single-rooted tree.*

Let $T = (V, E)$ be a connected, acyclic directed graph that expresses a hierarchical structure. The set of vertices V represents the entities that are being organized, and the set of edges E maps from a given entity to its respective superior.

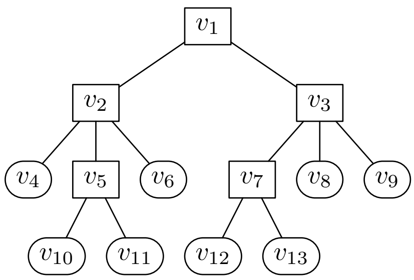

Figure 1: Hierarchy T

A vertex is said to have degree k if it has exactly k direct subordinates (incoming edges). Vertices with degree 0 are called leaves and vertices with degree > 0 are called nodes.

Definition. *The average degree of a hierarchy is the average degree of its nodes.*

If T is not empty, then there exists exactly one distinguished vertex $v_1 \in V$ that has no superior. All other vertices have exactly one superior.

For every $v \in V$ there exists a unique path that leads from v to v_1. The length of that path is referred to as depth of v. The depth of v_1 is 0.

Definition. *The average depth of a hierarchy is the average depth of its leaves.*

Let μ be the average degree of T, and let l be the average depth of T. Then there exists a perfectly regular tree T' in which *all* nodes have degree μ and *all* leaves are at depth l. Such a tree has μ^k vertices at any given depth k, which means that the total number of vertices is

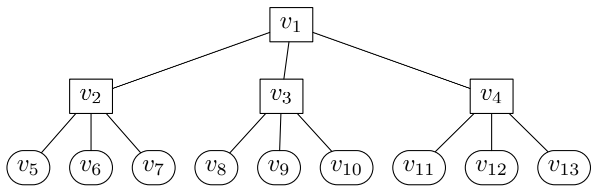

Figure 2: T approximated a by perfect 3-tree

$$n \;=\; \sum_{k=0}^{l} \mu^k \;=\; \frac{1-\mu^{l+1}}{1-\mu}.$$

Corollary. *In every perfectly regular hierarchy, the relationship between the average degree μ, the average depth l, and the number of vertices n is*

$$n\,\mu^{l+1} - \mu^l - n + 1 \;=\; 0.$$

Corollary. *The proportion of nodes in every perfectly regular hierarchy is*

$$\frac{1-\mu^l}{1-\mu^{l+1}}.$$

2 Application

Consider any given company hierarchy of $n = 10\,000$ employees. Nodes represent managers and leaves represent workers. Workers have an average chain of $l = 4$ superiors up to (and including) the big boss. If that hierarchy is somewhat regular, then approximately 10% of the workforce — 1 000 employees — would be required for purposes of management. The average team in that company would have approximately $\mu = 9,7$ members. If the company *doesn't* match these predictions, then its hierarchy would have to be somewhat irregular.